

Symmetric-Bounce Quantum State of the Universe ^{*}

Don N. Page [†]

Theoretical Physics Institute

Department of Physics, University of Alberta

Room 238 CEB, 11322 – 89 Avenue

Edmonton, Alberta, Canada T6G 2G7

(2009 August 14)

Abstract

A proposal is made for the quantum state of the universe that has an initial state that is macroscopically time symmetric about a homogeneous, isotropic bounce of extremal volume and that at that bounce is microscopically in the ground state for inhomogeneous and/or anisotropic perturbation modes. The coarse-grained entropy is minimum at the bounce and then grows during inflation as the modes become excited away from the bounce and interact (assuming the presence of an inflaton, and in the part of the quantum state in which the inflaton is initially large enough to drive inflation). The part of this pure quantum state that dominates for observations is well approximated by quantum processes occurring within a Lorentzian expanding macroscopic universe. Because this part of the quantum state has no negative Euclidean action, one can avoid the early-time Boltzmann brains and Boltzmann solar systems that appear to dominate observations in the Hartle-Hawking no-boundary wavefunction.

^{*}Alberta-Thy-09-09, arXiv:0907.1893

[†]Internet address: don@phys.ualberta.ca

Introduction

Even if physicists succeed in finding a so-called ‘Theory of Everything’ or TOE that gives the full set of dynamical laws for our universe, it appears that that will be insufficient to explain our past observations and to predict new ones. The reason is that each set of dynamical laws, at least of the kind we are familiar with, permits a wide variety of solutions, most of which would be inconsistent with our observations. We need a set of initial conditions and/or other boundary conditions to restrict the possible solutions to fit what we observe. In a quantum description of the universe with fixed dynamical laws (the analogue of the Schrödinger equation for nonrelativistic quantum mechanics), we need not only these dynamical laws but also the quantum state itself (cf. [1]). (We also need the rules for extracting observational probabilities from the quantum state [2, 3, 4, 5] for solving the measure problem in cosmology, which is another extremely important issue, but I shall not focus on that in this paper.)

To put it another way, our observations strongly suggest that our observed portion (or subuniverse [6] or bubble universe [7, 8] or pocket universe [9]) of the entire universe (or multiverse [10, 11, 12, 13, 14, 15, 16, 17] or metauniverse [18] or omnium [19] or megaverse [20]) is much more special than is implied purely by the known dynamical laws. For example, it is seen to be enormously larger than the Planck scale, with small large-scale curvature, and with approximate homogeneity and isotropy of the matter distribution on the largest scales that we can see today. It especially seems to have had extraordinarily high order in the early universe to enable its coarse-grained entropy to increase and to give us the observed second law of thermodynamics [21, 22, 23]. The known dynamical laws do not imply these observed conditions.

Leading proposals for special quantum states of the universe have been the Hartle-Hawking ‘no-boundary’ proposal [24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34] and the ‘tunneling’ proposals of Vilenkin, Linde, and others [35, 36, 37, 38, 39, 40]. In simplified toy models with a suitable inflaton, both of these classes of models have seemed to lead to the special observed features of our universe noted above.

However, Leonard Susskind [41] (cf. [42, 43, 44]) has made the argument, which I have elaborated [45], that in the no-boundary proposal the cosmological constant or quintessence or dark energy that is the source of the present observations of the cosmic acceleration [46, 47, 48, 49, 50, 51, 52] would give a very large Euclidean 4-hemisphere as an extremum of the Hartle-Hawking path integral that would apparently swamp the extremum from rapid early inflation by amplitude factors of the order of $e^{10^{122}}$. Therefore, to very high probability, the present universe should be very nearly empty de Sitter spacetime, which is certainly not what we observe. Even if we restrict to the very rare cases in which a solar system like ours occurs, the probability in the Hartle-Hawking no-boundary proposal seems to be much, much higher for a single solar system in an otherwise empty universe than for a solar system surrounded by other stars such as what we observe.

The tunneling proposals have also been criticized for various problems [53, 40, 54, 55, 56, 57]. For example, the main difference from the Hartle-Hawking no-

boundary proposal seems to be the sign of the Euclidean action [35, 36]. It then seems problematic to take the opposite sign for inhomogeneous and/or anisotropic perturbations without leading to some instabilities, and it is not clear how to give a sharp distinction between the modes that are supposed to have the reversed sign of the action and the modes that are supposed to retain the usual sign of the action. Vilenkin and his collaborators have emphasized [35, 39, 40] that the instabilities do not seem to apply to his particular tunneling proposal, which does not just reverse the sign of the Euclidean action. However, Vilenkin (with Garriga) admits [40] that “both wavefunctions are far from being rigorous mathematical objects with clearly specified calculational procedures. Except in the simplest models, the actual calculations of ψ_T and ψ_{HH} involve additional assumptions which appear reasonable, but are not really well justified.”

Therefore, at least unless and until any of these proposals can be made rigorous and can be shown conclusively to avoid the problems attributed to them, it is worth searching for and examining other possibilities for the quantum state of the universe or multiverse. In a previous paper [58], I proposed a ‘no-bang’ quantum state which is the equal mixture of the Giddings-Marolf states [59] that are asymptotically single de Sitter spacetimes in both past and future and are regular on the throat or neck of minimal three-volume. However, it does not appear to work if one adopts my proposal of volume averaging [2] to help solve the late-time aspect of the Boltzmann brain problem.

The Boltzmann brain problem [42, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 59, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84] is the problem that many cosmological theories seem to predict that our observations would be highly improbable in comparison with much more disordered observations of Boltzmann brains that these theories predict should enormously dominate over ordinary observers. Boltzmann brains are observers that appear from thermal or vacuum fluctuations. The probability of a Boltzmann brain per four-volume is extremely tiny (say roughly $e^{-10^{42}}$ [66, 68, 70]), but if the universe lasts for an infinite time, and especially if its three-volume grows asymptotically exponentially, and if there are only a finite number of ordinary observers per comoving three-volume, then per comoving volume the Boltzmann brains will dominate and make our ordered observations very atypical and improbable relative to the much more disordered typical Boltzmann brain observations. (The dominance by Boltzmann brains at very late times, which might occur in any universe that lasts forever, I call the late-time Boltzmann brain problem; the Hartle-Hawking no-boundary proposal appears to suffer from what might be called an early-time Boltzmann brain problem, that at all times Boltzmann brains seem to dominate over ordinary observers [42, 43, 44, 41, 45].)

Originally I proposed a solution to the Boltzmann brain problem in which the universe might be likely to decay before Boltzmann brains would dominate [62, 64, 66, 68, 70], but this seemed to require fine-tuning of whatever physics might determine the decay rate (though see [85] for a possible anthropic explanation of this decay rate). Therefore, I turned to another possible solution, that one should go from volume weighting to volume averaging [2] to extract observational probabilities. This would eliminate the effect of the exponentially growing 3-volumes in the asymptotic

future, though there still remains a much less rapid divergence on the weighting of Boltzmann brains from an infinite future lifetime of the universe, unless one went beyond 3-volume averaging to 4-volume averaging that would allow a possible anthropic explanation of a decaying universe [85]. However, if one goes from volume weighting to volume averaging to mitigate the late-time Boltzmann brain problem, the no-bang state then appears to suffer qualitatively from the same problem as the no-boundary state of being dominated by thermal perturbations of nearly empty de Sitter spacetime, so that almost all observers would presumably be Boltzmann brains. Since this would almost certainly make our observations very unlikely, the no-bang proposal apparently is observationally excluded if one uses volume averaging rather than volume weighting. (The no-boundary state appears to be excluded if either rule were used for extracting probabilities from the quantum state, since it has both an early-time and a late-time Boltzmann brain problem.)

In this paper, instead of the mixed ‘no-bang’ state, I shall propose a pure quantum state in which the Giddings-Marolf seed state [59] (before group averaging over diffeomorphisms) consists of quantum fluctuations about a uniform superposition of Lorentzian macroscopic components that are each time symmetric about a bounce of extremal 3-volume, with the quantum fluctuations being in their ground state at that moment of time symmetry for the macroscopic 4-geometry. With both signs of the Lorentzian time away from this momentarily-static bounce, the 3-volume will expand, typically in an inflationary manner if the matter is dominated by a sufficiently large homogeneous component of a scalar inflaton field. This inflationary expansion will then produce parametric amplification of the inhomogeneous and anisotropic modes in the usual manner to give density fluctuations at the end of inflation that then grow gravitationally to become nonlinear and produce the structure that we observe.

A slight aesthetic disadvantage of the symmetric-bounce quantum state in comparison with the no-boundary state is that in the symmetric-bounce proposal, the inhomogeneous fluctuations are put into their ground state at the bounce by a part of the proposal that is logically separate from the part of the proposal that gives the behavior of the homogeneous modes, whereas in the no-boundary proposal the behavior of both the inhomogeneous and homogeneous modes come out together from the same part of that proposal, that the histories that contribute to the path integral are regular on a complete complexified Euclidean manifold with no boundary other than the one on which the wavefunction is evaluated. However, this seems to be a small price to pay for avoiding the huge negative Euclidean actions of many nearly-empty de Sitter histories in the no-boundary proposal that make nearly empty spacetime much more probable than a nearly Friedmann-Robertson-Walker spacetime with high densities at early times that would fit our observations much better. To avoid making our observation of distant stars extremely improbable, as it appears to be in the no-boundary proposal, it seems well worth giving up the simple no-boundary unified description of the behavior of both the homogeneous inflationary modes and the inhomogeneous fluctuation modes.

1 Homogeneous modes with an inflaton and a cosmological constant

First, let us focus on the behavior of the homogeneous, isotropic modes of the symmetric-bounce quantum seed state. That is, take each quasiclassical component of the macroscopic spacetime geometry, without the quantum fluctuations, to be a Friedmann-Robertson-Walker (FRW) model driven by homogeneous matter fields. For concreteness and simplicity, consider the case of a positive cosmological constant $\Lambda = 3/b^2$ and a single inflaton that is a homogeneous free scalar field $\phi(t)$ of mass m , and take the FRW model to be $k = +1$ so that the spatial sections are homogeneous, isotropic 3-spheres of radius $a(t)$. Then the macroscopic spacetime metric can be taken to be

$$ds^2 = -N^2 dt^2 + a^2(t) d\Omega_3^2. \quad (1)$$

Using units in which $\hbar = c = 1$, but writing G explicitly, one can write the Lorentzian action as (cf. [86])

$$\begin{aligned} S &= \int N dt 2\pi^2 a^3 \left\{ \frac{3}{8\pi G} \left[-\left(\frac{1}{Na} \frac{da}{dt} \right)^2 + \frac{1}{a^2} - \frac{\Lambda}{3} \right] + \frac{1}{2} \left(\frac{1}{N} \frac{d\phi}{dt} \right)^2 - \frac{1}{2} m^2 \phi^2 \right\} \\ &= \frac{3\pi}{4G} \int N dt a^3 \left\{ -\left(\frac{1}{Na} \frac{da}{dt} \right)^2 + \left(\frac{1}{N} \frac{d\phi}{dt} \right)^2 + \frac{1}{a^2} - \frac{1}{b^2} - m^2 \phi^2 \right\} \\ &= \frac{3\pi}{4Gm^2} \int n dt r^3 \left\{ -\left(\frac{1}{nr} \frac{dr}{dt} \right)^2 + \left(\frac{1}{n} \frac{d\phi}{dt} \right)^2 + \frac{1}{r^2} - \lambda - \phi^2 \right\} \\ &= \frac{3\pi}{4} \frac{m_{\text{Pl}}^2}{m^2} \int n dt e^{3\alpha} \left[-n^{-2} (\dot{\alpha}^2 - \dot{\phi}^2) + e^{-2\alpha} - \lambda - \phi^2 \right] \\ &= \frac{1}{2} \int n dt \left[\left(\frac{1}{n} \frac{ds}{dt} \right)^2 - V \right] = \frac{1}{2} \int dt \left[\frac{1}{\nu} \left(\frac{d\hat{s}}{dt} \right)^2 - \nu \right], \end{aligned} \quad (2)$$

where $b \equiv \sqrt{3/\Lambda}$ is the radius of the throat of pure de Sitter with the same value of the cosmological constant, $n \equiv mN$ is a rescaled lapse function that is dimensionless if t is taken to be dimensionless, $\lambda \equiv \Lambda/(3m^2) \equiv 1/(mb)^2$ is a dimensionless measure of the cosmological constant in units given by the mass of the inflaton, $r \equiv e^\alpha \equiv ma$ and $\phi \equiv \sqrt{4\pi G/3} \phi$ are dimensionless forms of the scale factor and inflaton scalar field (leaving $G \equiv m_{\text{Pl}}^{-2}$ to have the dimensions of inverse mass squared or of area), an overdot represents a derivative with respect to t , the DeWitt metric [87] on the minisuperspace is

$$ds^2 = \frac{3\pi}{2Gm^2} e^{3\alpha} (-d\alpha^2 + d\phi^2), \quad (3)$$

the ‘potential’ on the minisuperspace is

$$V = \frac{3\pi}{2Gm^2} e^{3\alpha} (\phi^2 + \lambda - e^{-2\alpha}), \quad (4)$$

the rescaled lapse function is $\nu \equiv nV = mNV$, and the conformal minisuperspace metric is

$$d\hat{s}^2 = Vds^2 = \left(\frac{3\pi}{2Gm^2}\right)^2 e^{6\alpha}(\varphi^2 + \lambda - e^{-2\alpha})(-d\alpha^2 + d\varphi^2). \quad (5)$$

To get some reasonable numbers for the dimensionless constants in these equations, take $\Omega_\Lambda = 0.72 \pm 0.04$ from the third-year WMAP results of [50] and $H_0 = 72 \pm 8$ km/s/Mpc from the Hubble Space Telescope key project [88], and drop the error uncertainties to get $G\Lambda = 3\Omega_\Lambda GH_0^2 \approx 3.4 \times 10^{-122}$, which would give $b = \sqrt{3/\Lambda} \approx 9.4 \times 10^{60} \sqrt{G}$. Then use the estimate that $m \approx 1.5 \times 10^{-6} G^{-1/2} \approx 7.5 \times 10^{-6} (8\pi G)^{-1/2}$ [89, 90] from the measured fluctuations of the cosmic microwave background to get that the prefactor of the action is $(3\pi/4)(m_{\text{Pl}}/m)^2 \approx 1.0 \times 10^{12}$, and the dimensionless measure of the cosmological constant is $\lambda \equiv \Lambda/(3m^2) \equiv 1/(mb)^2 \approx 5.0 \times 10^{-111}$. Thus λ may be taken to be extremely tiny, and for histories in which α and/or φ are of the order of unity or greater, the action will be very large and so should give essentially classical behavior, at least for the homogeneous, isotropic part of the geometry.

The constraint equation and independent equation of motion can now be written as

$$\begin{aligned} \left(\frac{1}{Na} \frac{da}{dt}\right)^2 &= \left(\frac{1}{N} \frac{d\varphi}{dt}\right)^2 + m^2 \varphi^2 + \frac{1}{b^2} - \frac{1}{a^2}, \\ \frac{1}{N} \frac{d}{dt} \left(\frac{1}{N} \frac{d\varphi}{dt}\right) + \left(\frac{3}{Na} \frac{da}{dt}\right) \left(\frac{1}{N} \frac{d\varphi}{dt}\right) + m^2 \varphi^2 &= 0, \end{aligned} \quad (6)$$

for general lapse function from the second form of the action above,

$$\begin{aligned} \dot{r}^2 &= r^2(\dot{\varphi}^2 + \varphi^2 + \lambda) - 1, \\ \ddot{\varphi} + 3\frac{\dot{r}}{r}\dot{\varphi} + \varphi &= 0, \end{aligned} \quad (7)$$

from the third form of the action with $n = 1$, and

$$\begin{aligned} \dot{\alpha}^2 - \dot{\varphi}^2 &= \varphi^2 + \lambda - e^{-2\alpha}, \\ \ddot{\varphi} + 3\dot{\alpha}\dot{\varphi} + \varphi &= 0, \end{aligned} \quad (8)$$

for the fourth form of the action above with $n = 1$, which will henceforth be assumed.

Although it is a redundant equation, one may readily derive from Eqs. (8) that

$$\ddot{\alpha} = e^{-2\alpha} - 3\dot{\varphi}^2 \quad (9)$$

when $n = 1$. Then when neither side of the constraint (first) equation part of Eqs. (8) vanishes (e.g., when $V \neq 0$), and when $\dot{\varphi} \neq 0$, one may define $f' \equiv df/d\varphi = \dot{f}/\dot{\varphi}$ and reduce Eqs. (8) to the single second-order differential equation (cf. [86])

$$\alpha'' = \frac{(\alpha'^2 - 1)(\varphi\alpha' + 3\varphi^2 + 3\lambda - 2e^{-2\alpha})}{\varphi^2 + \lambda - e^{-2\alpha}}. \quad (10)$$

Alternatively, when $V \neq 0$ (or equivalently $\dot{\alpha}^2 \neq \dot{\varphi}^2$), but when $\dot{\alpha} \neq 0$ instead of $\dot{\varphi} \neq 0$, one can write

$$\frac{d^2\varphi}{d\alpha^2} = \frac{(d\varphi/d\alpha)^2 - 1}{\varphi^2 + \lambda - e^{-2\alpha}} \left[(3\varphi^2 + 3\lambda - 2e^{-2\alpha}) \frac{d\varphi}{d\alpha} + \varphi \right]. \quad (11)$$

Yet another way to get the equations of motion is to note that the fifth form of the action from Eq. (2) gives the trajectories of a particle of mass-squared V in the DeWitt minisuperspace metric [87] ds^2 , and the sixth form of the action gives timelike geodesics in the conformal minisuperspace metric $d\hat{s}^2 = Vds^2$. When one goes to the gauge $\nu = 1$, then $(d\hat{s}/dt)^2 = -1$, so that along the classical timelike geodesics of $d\hat{s}^2$, the Lorentzian action is $S = -\int dt = -\int \sqrt{-d\hat{s}^2}$, minus the proper time along the timelike geodesic of $d\hat{s}^2$. However, one must note that the conformal metric $d\hat{s}^2 = Vds^2$ is singular at $V = 0$, that is at $\varphi^2 + \lambda = e^{-2\alpha} \equiv 1/(ma)^2$, whereas there is no singularity in the DeWitt metric ds^2 or the spacetime metric along this hypersurface (line) in the two-dimensional minisuperspace (α, φ) under consideration. The second-order differential equations (10) and (11) also break down at $V = 0$ and must be supplemented by the continuity of $\dot{\alpha}$ and of $\dot{\varphi}$ (in a gauge in which $n \neq 0$ is continuous there) across the $V = 0$ hypersurface (line).

2 Symmetric-bounce proposal for the homogeneous modes

My symmetric-bounce proposal for the homogeneous modes, which are represented classically by the trajectories in the (α, φ) minisuperspace, is that one takes the set of all Lorentzian symmetric bounce trajectories, those that have $\dot{\alpha} = \dot{\varphi} = 0$ somewhere along the classical trajectory. By the definition Eq. (4) of the potential $V(\alpha, \varphi)$ and by the constraint Eq. (8), this point of the trajectory will have $V = 0$ or $a = \sqrt{3}/\sqrt{4\pi G m^2 \phi^2 + \Lambda}$ or

$$\alpha = \alpha_{\text{bounce}}(\varphi) \equiv -\frac{1}{2} \ln(\varphi^2 + \lambda). \quad (12)$$

The classical trajectory that has $\dot{\alpha} = \dot{\varphi} = 0$ at $(\alpha, \varphi) = (\alpha_{\text{bounce}}(\varphi_b), \varphi_b)$ for some value of $\varphi_b \equiv \varphi_{\text{bounce}}$ will be time symmetric about this bounce point, so if one sets $t = 0$ there and uses a time-symmetric lapse function, $n(t) = n(-t)$, then $(\alpha(t), \varphi(t)) = (\alpha(-t), \varphi(-t))$.

A generic trajectory in the (α, φ) minisuperspace can be labeled by the location at which it crosses some hypersurface (e.g., at its value of φ on a hypersurface of fixed α) and by its direction there (e.g., its value of $\alpha' = d\alpha/d\varphi$), since once the direction is fixed, the constraint equation determines the values of both $\dot{\alpha}$ and of $\dot{\varphi}$. Thus the generic minisuperspace trajectories form a two-parameter family. However, the symmetric-bounce trajectories may be labeled by the single parameter φ_b of the value of φ that it has on the hypersurface $\alpha = \alpha_{\text{bounce}}(\varphi)$, since at that point on a symmetric-bounce trajectory, the values of $\dot{\alpha}$ and of $\dot{\varphi}$ are both determined to

be zero. Therefore, in terms of the classical measure [91] on the two-dimensional space of minisuperspace trajectories, the symmetric-bounce trajectories are a set of measure zero. This restriction on the classical phase space of trajectories is precisely analogous to the restriction of the no-boundary state on the set of classical trajectories [34], though the details of the restriction are slightly different (precisely real classical trajectories that have symmetric bounces for the symmetric-bounce state).

However, since I am proposing that the quantum state is a superposition of initially quasiclassical components that give a one-parameter set of classical trajectories, to make the proposal definite I do need to give the coefficients in the quantum quantum superposition or the measure for the classical trajectories, analogous to the weighting by the exponential of minus the (negative) Euclidean action for the no-boundary proposal and by essentially the exponential of the Euclidean action for the tunneling proposal. I shall propose that the one-parameter set of classical trajectories are uniformly distributed over the symmetric-bounce hypersurface $(\alpha_{\text{bounce}}(\varphi_b), \varphi_b)$, with no weighting by the exponential of either minus or plus the Euclidean action. Thus my symmetric-bounce quantum state has a measure that is basically the geometric mean of the no-boundary and tunneling proposals. For such a uniform measure, $\mu(\varphi_b)d\varphi_b$, I shall take the magnitude of the metric induced on this hypersurface by the DeWitt minisuperspace metric [87] given by Eq. (3), after dropping the constant factor $3\pi/(2Gm^2)$. That is, I shall take

$$\begin{aligned}\mu(\varphi_b)d\varphi_b &= \sqrt{2Gm^2/(3\pi)} |ds| \\ &= e^{3\alpha_{\text{bounce}}(\varphi_b)/2} \sqrt{1 - [d\alpha_{\text{bounce}}(\varphi_b)/d\varphi_b]^2} d\varphi_b \\ &= (\varphi_b^2 + \lambda)^{-7/4} \sqrt{|\varphi_b^2 + \varphi_b + \lambda| |\varphi_b^2 - \varphi_b + \lambda|} d\varphi_b.\end{aligned}\tag{13}$$

The coefficients in the continuum quantum superposition I shall take to be the real positive square roots of this measure. I should like to emphasize that, like all other proposals for the quantum state of the universe, this is just a proposal and is not derived from previously accepted principles.

The symmetric-bounce proposal specifies the form of the quantum state at the bounce, but, unlike some other proposals such as the symmetric initial condition [92], it does not impose any requirement that the wavefunction be normalizable over the entire superspace. Indeed, even for the minisuperspace of the homogeneous isotropic modes of the scale factor variable α and the inflaton field variable φ , the symmetric-bounce wavefunction propagates unabated to arbitrarily large α and so is not normalizable, that is, it is not square-integrable over the (α, φ) space with the area element induced from the DeWitt metric [87].

Because the symmetric-bounce hypersurface $(\alpha_{\text{bounce}}(\varphi_b), \varphi_b)$ becomes asymptotically null sufficiently rapidly with $|\varphi_b|$ for large $|\varphi_b|$, so that $\mu(\varphi_b) \sim |\varphi_b|^{-3/2}$ for large $|\varphi_b|$, the total measure $\mu(\varphi_b)d\varphi_b$ integrated over all φ_b from minus infinity to plus infinity is finite. It is dominated by the regions where $\varphi_b^2 \sim \lambda$, giving $\int_{-\infty}^{\infty} \mu(\varphi_b)d\varphi_b \approx (4/3)\lambda^{-3/4} \approx 7 \times 10^{82}$ for $\lambda \approx 5.0 \times 10^{-111}$ as estimated above. Here I shall ignore one-loop quantum corrections [93, 94, 95], partly because of the fact

that if they are important, unknown higher-loop effects are likely also to be important. Such quantum corrections should be unimportant when the energy density is much less than the Planck density, e.g., for $\varphi^2 \ll G^{-1}m^{-2} \sim 10^{12}$. The energy density at the bounce is less than the Planck value for over 99.9% of the measure of the symmetric bounce trajectories with $\varphi_b^2 > 1$.

The symmetric-bounce homogeneous spacetimes, labeled by the value of φ_b where each of them has its symmetric bounce on the symmetric-bounce hypersurface $(\alpha_{\text{bounce}}(\varphi_b), \varphi_b)$, may be divided into five classes depending on which spacelike or timelike segment of the symmetric-bounce hypersurface at which each of them has its symmetric bounce. These segments are divided by the points at which the symmetric-bounce hypersurface becomes null in the DeWitt metric of Eq. (3) and crosses from being spacelike to timelike or from timelike to spacelike. These points are where $1 - [d\alpha_{\text{bounce}}(\varphi_b)/d\varphi_b]^2 = 0$ or $(\varphi_b^2 + \varphi_b + \lambda)(\varphi_b^2 - \varphi_b + \lambda) \equiv (\varphi_b + \varphi_2)(\varphi_b + \varphi_1)(\varphi_b - \varphi_1)(\varphi_b - \varphi_2) = 0$, or at $\varphi_b = -\varphi_2$, $\varphi_b = -\varphi_1$, $\varphi_b = +\varphi_1$, and $\varphi_b = +\varphi_2$, where $\varphi_1 = (1/2)(1 - \sqrt{1 - 4\lambda}) \approx \lambda$ and $\varphi_2 = (1/2)(1 + \sqrt{1 - 4\lambda}) \approx 1$. Then one may define Segment 1 to be the spacelike part of the symmetric-bounce hypersurface with $\varphi_b < -\varphi_2$, Segment 2 to be the timelike part with $-\varphi_2 < \varphi_b < -\varphi_1$, Segment 3 to be the spacelike part with $-\varphi_1 < \varphi_b < \varphi_1$, Segment 4 to be the timelike segment with $\varphi_1 < \varphi_b < \varphi_2$, and Segment 5 to be the spacelike segment with $\varphi_2 < \varphi_b$. Under the symmetry $\varphi \rightarrow -\varphi$, Segments 1 and 5 are interchanged, Segments 2 and 4 are interchanged, and Segment 3 is interchanged with itself. Therefore, without loss of generality, one may take $\varphi_b \geq 0$ and consider only Segments 3, 4, and 5. One may estimate that for $\lambda \approx 5.0 \times 10^{-111}$, Segments 1 and 5 each have measure $\approx (1/2)B(1/4, 3/2) \approx 1.748$, Segments 2 and 4 each have measure $\approx (2/3)\lambda^{-3/4} \approx 3.5 \times 10^{82}$, and Segment 3 has measure $\approx (\pi/2)\lambda^{1/4} \approx 4.2 \times 10^{-28}$.

At a symmetric bounce, using the gauge $n = 1$, one has $\dot{\alpha} = \dot{\varphi} = 0$, but $\ddot{\alpha} = e^{-2\alpha_{\text{bounce}}(\varphi_b)} = \varphi_b^2 + \lambda$ and $\ddot{\varphi} = -\varphi_b$, so the trajectory starts with the slope $d\alpha/d\varphi = \ddot{\alpha}/\ddot{\varphi} = -(\varphi_b^2 + \lambda)/\varphi_b = d\varphi_b/d\alpha_{\text{bounce}}(\varphi_b)$, orthogonal to the symmetric-bounce hypersurface in the DeWitt metric of Eq. (3). As one moves slightly away from the symmetric bounce, φ always starts evolving toward zero, and α always starts evolving toward larger values. For a symmetric bounce in Segments 1, 3, and 5, the trajectory initially moves into the minisuperspace region above the symmetric-bounce hypersurface, $\alpha > \alpha_{\text{bounce}}(\varphi)$, and is there timelike in the DeWitt metric ($d\alpha^2 > d\varphi^2$); for a symmetric bounce in Segments 2 and 4, the trajectory initially moves into the minisuperspace region below the symmetric-bounce hypersurface, $\alpha < \alpha_{\text{bounce}}(\varphi)$, and is there spacelike in the DeWitt metric ($d\alpha^2 < d\varphi^2$).

Symmetric-bounce homogeneous spacetimes that bounce on Segment 3 thereafter move along timelike trajectories ever upward in the (α, φ) minisuperspace and hence expand forever. Their dynamics are always dominated by the positive cosmological constant and behave very nearly like empty de Sitter universes. In my proposed measure, their measure is only $\sim 10^{-28}$ that of Segments 1 and 5 and only $\sim 10^{-110}$ that of Segments 2 and 4, so these nearly empty spacetimes do not seem to contribute much to the measure for observations, unlike their contribution to the Hartle-Hawking no-boundary quantum state [41, 42, 43, 44, 45].

Symmetric-bounce spacetimes that bounce on Segment 2 or 4, with $\lambda^2 \approx \varphi_1^2 <$

$\varphi_b^2 < \varphi_2^2 \approx 1$, except for φ_b^2 sufficiently close to 1, generally have a period of expansion during which the scalar field oscillates rapidly relative to the expansion. When averaged over each oscillation, the mean value of $\dot{\varphi}^2$ is nearly the same as that of φ^2 (in a gauge with $n = 1$, which I shall assume unless stated otherwise), which is equivalent to saying that the pressure exerted by the scalar inflaton averages to near zero over each oscillation. Then the scalar field acts essentially like pressureless dust, with a total rationalized dimensionless ‘mass’ that is nearly constant:

$$M \equiv (\varphi^2 + \dot{\varphi}^2)r^3 = (\varphi^2 + \dot{\varphi}^2)e^{3\alpha} = \frac{8\pi G}{3}m\rho a^3, \quad (14)$$

where $a = r/m$ is the physical scale factor and

$$\rho = \frac{1}{2} \left[m^2 \phi^2 + \left(\frac{1}{N} \frac{d\phi}{dt} \right)^2 \right] = \frac{3m^2}{8\pi G}(\varphi^2 + \dot{\varphi}^2) \quad (15)$$

is the energy density of the scalar field with our choice of $n = mN = 1$ to make our time coordinate t dimensionless (and with d/dt being denoted by an overdot). Thus the dimensionless M is $4Gm/(3\pi)$ times the integral of the energy density ρ over the volume $2\pi^2 a^3$ of the 3-sphere of physical scale factor a and of dimensionless scale factor $r \equiv e^\alpha \equiv ma$. The approximate constancy of M during the ‘dust’ regime results from the fact that the integral of

$$\frac{dM}{d\alpha} = 3(\varphi^2 - \dot{\varphi}^2)e^{3\alpha} \quad (16)$$

is approximately zero over each oscillation of the scalar field.

Then during such a ‘dust’ phase, the dimensionless scale factor $r = ma$ evolves according to

$$\dot{r}^2 = \lambda r^2 + \frac{M}{r} - 1 \quad (17)$$

with M very nearly constant. As a function of the dimensionless scale factor $r \equiv ma$ at fixed M , the right hand side has a minimum at $r = [M/(2\lambda)]^{1/3}$ that is positive if $27\lambda M^2 > 4$, so when this condition holds, the universe will expand forever from any initial r if M stays constant. However, this sufficient (but not necessary) condition for expansion forever does not hold for any $\varphi_b^2 \ll 1$ for which M stays nearly constant after the bounce, at which one has

$$r_b = \frac{1}{\sqrt{\varphi_b^2 + \lambda}}, \quad M_b = \frac{\varphi_b^2}{(\varphi_b^2 + \lambda)^{3/2}}, \quad (18)$$

since obviously the right hand side of Eq. (17) is zero at the bounce.

That is, although $27\lambda M^2 > 4$ with constant M is sufficient for the universe to expand forever in our simple $k = +1$ FRW model with a cosmological constant and a massive scalar field that acts like dust, it is not necessary. Conversely, $27\lambda M^2 < 4$ is necessary but not sufficient for recollapse. If $27\lambda M^2 < 4$ does hold, one also needs

that r be at an allowed value (one giving $\dot{r}^2 \geq 0$) less than the minimum of the right hand side of Eq. (17), which is equivalent to $2\lambda r^3 < M$. Thus this model $k = +1$ FRW Λ -dust model will recollapse (assuming M stays constant) if and only if

$$\begin{aligned} 2\lambda r^3 < M < \frac{2}{\sqrt{27\lambda}} &\Leftrightarrow \\ 27\lambda^3 r^6 < 6.75\lambda M^2 < 1. \end{aligned} \quad (19)$$

Using Eq. (18), which leads to a nearly constant $M \approx M_b$ when $\varphi_b^2 \ll 1$, we see that our $k = +1$ FRW Λ -scalar model with the symmetric-bounce initial condition will recollapse if and only if

$$10^{-110} \approx 2\lambda \lesssim \varphi_b^2 \lesssim O(1). \quad (20)$$

This is the part of Segments 2 and 4 with larger values of φ_b^2 , plus a bit into Segments 1 and 5. For $\lambda \ll \varphi_b^2$, well into the interior of this open set of values of φ_b , the evolution will have $\lambda r^2 \ll M/r$ during the evolution, so the dimensionless collapse time Δt with $n = 1$ will be approximately $(\pi/2)r_b \approx \pi/(2\varphi_b)$. For φ_b^2 large enough to give a density sufficient for nucleosynthesis (e.g., at the density our universe had at an age of a few minutes), the lifetime in proper time would be of the order of minutes, far too short for the evolution of stars and observers that depend upon stars. Although Segments 2 and 4 dominate the measure given by Eq. (13) by factors of the order of 10^{82} , they do not do so by factors anywhere near the inverses (say $\sim e^{10^{42}}$) of the exponentially tiny relative probabilities of forming Boltzmann brains, so the resulting symmetric-bounce universes will presumably have extremely tiny probabilities for observers and should contribute negligibly to observational probabilities. (This is unlike the case of the Hartle-Hawking no-boundary proposal, where factors from the negative Euclidean action, say $\sim e^{10^{122}}$, *can* be much greater than the inverses of the relative probabilities to form Boltzmann brains or even Boltzmann solar systems.)

The symmetric-bounce initial conditions that lead to recollapse actually extend past $\varphi_b^2 = \varphi_2^2 \approx 1$ into Segments 1 and 5, but there the dust approximation that M is nearly constant breaks down. It is difficult to give a good approximate closed-form treatment for $\varphi_b^2 \sim 1$, but for φ_b^2 a few times unity, one enters the slow-roll inflationary regime where M grows greatly during a period of inflation that can be estimated fairly accurately under the approximation that $\varphi_b^2 \gg 1$.

3 Approximate solutions for the inflationary regime

Let us now focus on the regime in which the initial (at the bounce) value of $\varphi^2 \equiv 4\pi G\phi^2/3$, that is φ_b^2 , is at least somewhat large compared to unity, so that the evolution away from the symmetric bounce starts with a period of slow-roll inflation that includes at least several e-folds of expansion. In this Section, we want to set up some theoretical analysis before turning in the next Section to a numerical calculation of how many e-folds of inflation occur, as a function of φ_b , and also of

the φ_b -dependence of the asymptotic value, in the ‘dust’ regime that follows the inflationary regime, of the total rationalized dimensionless ‘mass’ M given by Eq. (14).

Without loss of generality, assume that the value of φ at the bounce, φ_b , is positive, so when it is greater than $\varphi_2 \approx 1$, the FRW spacetime starts on Segment 5 with $r_b \approx 1/\varphi_b$. During the slow-roll inflationary regime with $\varphi \gg 1$, we have $\varphi^2 \gg \dot{\varphi}^2$. Since during inflation we have $\dot{\varphi}^2 + \varphi^2 \gtrsim 1 \gg \lambda$, we can neglect the cosmological constant term λ during inflation and take the inflationary equations to be Eqs. (7) or (8) with λ dropped:

$$\begin{aligned} \dot{r}^2 &= r^2(\varphi^2 + \dot{\varphi}^2) - 1, \\ \ddot{\varphi} + 3\frac{\dot{r}}{r}\dot{\varphi} + \varphi &= 0, \end{aligned} \tag{21}$$

in terms of the dimensionless scale factor $r \equiv e^\alpha \equiv ma$, or

$$\begin{aligned} \dot{\alpha}^2 &= \varphi^2 + \dot{\varphi}^2 - e^{-2\alpha}, \\ \ddot{\varphi} + 3\dot{\alpha}\dot{\varphi} + \varphi &= 0, \end{aligned} \tag{22}$$

in term of the logarithm $\alpha = \ln(ma)$ of the scale factor, and in terms of the dimensionless form $\varphi \equiv \sqrt{4\pi G/3}\phi$ of the inflaton scalar field ϕ .

From Eqs. (21), one can readily derive, as an alternative to the redundant Eq. (9) when λ is neglected, that

$$\ddot{r} = r(\varphi^2 - 2\dot{\varphi}^2). \tag{23}$$

We shall define the inflationary period as that first period immediately after the symmetric bounce when λ is negligible (so as not to consider inflation by the cosmological constant) and when $\ddot{r} > 0$, so that the scale factor of the universe is accelerating with respect to cosmic proper time. This is equivalent, with λ negligible, to the first period during which $2\dot{\varphi}^2 < \varphi^2$. Let us define N (or $N(\varphi_b)$, since it depends on the initial value φ_b at the bounce) to be the number of e-folds of the inflationary period, the change in the logarithm α of the scale factor during the inflationary period that starts with $\varphi = \varphi_b > 0$ and $\dot{\varphi} = 0$ at $\alpha = \alpha_b(\varphi_b) \equiv \alpha_{\text{bounce}}(\varphi_b) = -\ln \varphi_b$ by Eq. (12) with λ neglected and that ends at $\alpha = \alpha_e(\varphi_b)$ where φ has first dropped to the then-positive value of $-\sqrt{2}\dot{\varphi}$:

$$N(\varphi_b) \equiv \alpha_e(\varphi_b) - \alpha_b(\varphi_b). \tag{24}$$

It also is convenient to define a shifted scale-factor logarithm

$$\beta \equiv \alpha - \alpha_b \equiv \alpha + \ln \varphi_b, \tag{25}$$

which increases monotonically from $\beta_b = 0$ at the bounce to $\beta_e = N$ at the end of the inflationary period. Then the φ_b -dependent number of e-folds of inflation may be defined to be $N(\varphi_b) = \beta_e(\varphi_b)$. $N(\varphi_b)$ will be large if $\varphi_b \gg 1$, which is what we

shall assume, though many of the results below turn out to be quite accurate even if φ_b is as small as 3.

Now I shall give a sequence of increasingly better approximations for the early phase of inflation, followed by numerical calculations of $N(\varphi_b)$ and of the aftermath of inflation, such as the asymptotic value of the total rationalized dimensionless ‘mass’ M given by Eq. (14).

The simplest approximation is for the period when φ remains very nearly the same as its initial value φ_b and when $\dot{\varphi}$ is negligible in comparison. Then the first of Eqs. (21) becomes $\dot{r}^2 \approx r^2 \varphi_b^2 - 1$, with the solution

$$r \approx \varphi_b^{-1} \cosh \varphi_b t, \quad (26)$$

which gives de Sitter spacetime at this level of approximation. However, this level of approximation does not remain good indefinitely, since the second of Eqs. (21) implies that φ gradually decreases.

For $\varphi_b t \gg 1$ but still $\varphi^2 \gg \dot{\varphi}^2$ (so that several e-folds of inflation have occurred but one is not yet near the end of inflation), one is in the flat ($e^{-2\alpha} \ll \varphi^2 + \dot{\varphi}^2$) slow-roll ($\dot{\varphi}^2 \ll \varphi^2$) regime where the first of Eqs. (21) or (22) now becomes $\dot{r} \approx r\varphi$ or $\dot{\alpha} \approx \varphi$, so that the second of Eqs. (21) or (22) becomes $\ddot{\varphi} + 3\varphi\dot{\varphi} + \varphi \approx 0$, which has the attractor solution [96]

$$\varphi = \text{const.} - t/3 \sim \varphi_b - t/3. \quad (27)$$

Then one gets

$$\alpha \approx \text{const.}' + (\text{const.})t - t^2/6 \sim \alpha_b + \varphi_b t - t^2/6 \sim \alpha_b + 1.5(\varphi_b^2 - \varphi^2). \quad (28)$$

Since inflation ends when φ drops down to $-\sqrt{2}\dot{\varphi}$, which by the slow-roll approximation (no longer valid near the end of inflation but giving the right order of magnitude) is $\sqrt{2}/3$, which is much less than φ_b that we are assuming is much larger than unity, we get as the leading approximation for the number of e-foldings of inflation that $N(\varphi_b) \sim 1.5\varphi_b^2$. However, we shall find below that there is also a term logarithmic in φ_b , as well as terms that are inverse powers of φ_b^2 , plus a constant term that may be evaluated numerically.

If one looks at just the flat regime where $r^2(\varphi^2 + \dot{\varphi}^2) \gg 1$ but does not impose the slow-roll condition $\dot{\varphi}^2 \ll \varphi^2$, one can see that Eq. (10) with $U \equiv -\alpha' \equiv -d\alpha/d\varphi$ becomes the autonomous first-order differential equation

$$\frac{dU}{d\varphi} = (U^2 - 1)\left(\frac{U}{\varphi} - 3\right). \quad (29)$$

During slow-roll inflation with $\varphi \gg 1$, the solution will exponentially rapidly approach the attractor solution

$$U = 3\varphi + \frac{1}{3\varphi} - \frac{2}{27\varphi^3} + \frac{11}{243\varphi^5} - \frac{10}{243\varphi^7} + O(\varphi^{-9}). \quad (30)$$

This then gives

$$\alpha \approx \text{const.} - \frac{3}{2}\varphi^2 - \frac{1}{3}\ln \varphi - \frac{1}{27\varphi^2} + \frac{11}{972\varphi^4} - \frac{5}{729\varphi^6} + O(\varphi^{-8}), \quad (31)$$

where the const. term depends upon φ_b . One can see that this formula leads to a $(1/3)\ln \varphi_b$ term in $N(\varphi_b)$, but the value of the constant term in $N(\varphi_b)$ and of the terms that go as inverse powers of φ_b^2 require the behavior both before the entry into the flat regime and after the exit from the slow-roll regime.

Next, let us go to a better approximation during the first stages of inflation, not assuming one has entered the flat regime where the spatial curvature term $e^{-2\alpha}$ may be neglected. If one inserts the approximate solution for $r(t)$ from Eq. (26) into the second one of Eqs. (21) and solves it to the leading nontrivial order in $1/\varphi_b$, one gets the better approximation for the scalar field that is (cf. [97])

$$\varphi \approx \varphi_b - \frac{1}{3\varphi_b} [\ln \cosh(\varphi_b t) + \tanh^2(\varphi_b t)]. \quad (32)$$

Analogously, if one inserts the approximate solution for $\varphi(t)$ from Eq. (27) into the first one of Eqs. (21) and solves it under the slow-roll approximation, one gets the better approximation for the dimensionless scale factor $r = ma$ that is

$$r \equiv e^\alpha \approx \varphi_b^{-1} \cosh(\varphi_b t - t^2/6). \quad (33)$$

Both of these approximations are valid for all $t \ll \varphi_b$, both the regime in which the spatial curvature is not negligible and the early stages of the slow-roll regime in which φ has not rolled down very close to the bottom. One might have thought it would be yet a better improvement to take the argument of the hyperbolic functions in the expression for φ to be the same as they are given in the hyperbolic functions in the expression for r , namely $\varphi_b t - t^2/6$, but this would invalidate the fact that during the entire flat slow-roll regime, $\dot{\varphi}$ stays very close to $-1/3$. For $1 \ll \varphi_b t \ll \varphi_b^2$, so that one is in the early part of the flat slow-roll regime, one has

$$\varphi \approx \varphi_b - \frac{1}{3}t - \frac{1 - \ln 2}{3\varphi_b} \quad (34)$$

and

$$\alpha \equiv \ln r \approx \varphi_b t - \frac{1}{6}t^2 - \ln(2\varphi_b) \approx \frac{3}{2}\varphi_b^2 - \ln \varphi_b - \frac{3}{2}\varphi^2 - (1 - \ln 2)\frac{\varphi}{\varphi_b} - \ln 2. \quad (35)$$

For an even better approximation during the early stages of the slow-roll regime, one can use Eq. (11) and the definition $R \equiv e^\beta = r/r_b = \varphi_b r = \varphi_b ma$ to get

$$\begin{aligned} \varphi \approx & \varphi_b - \frac{1}{3\varphi_b} \left[\ln R + 1 - \frac{1}{R^2} \right] \\ & - \frac{1}{162\varphi_b^3} \left[9 \ln^2 R + 12 \ln R - \frac{54}{R^2} \ln R \right. \\ & + 18 \left(\arccos \frac{1}{R} \right)^2 + 36 \left(\arccos \frac{1}{R} \right) \frac{\sqrt{R^2 - 1}}{R^2} \\ & \left. - 16 + \frac{3}{R^2} + \frac{9}{R^4} + \frac{4}{R^6} \right] + O(\varphi_b^{-5}). \end{aligned} \quad (36)$$

Taking this expression into the flat regime for which $\beta \equiv \ln R \gg 1$ gives

$$\varphi \approx \varphi_b - \frac{\beta + 1}{3\varphi_b} - \frac{9\beta^2 + 12\beta + 4.5\pi^2 - 16}{162\varphi_b^3} + O(\varphi_b^{-5}). \quad (37)$$

When this approximation for $\varphi(\beta)$ in the flat slow-roll regime is inverted and matched to Eq. (31), one gets

$$\alpha \approx \frac{3}{2}\varphi_b^2 - \frac{2}{3}\ln \varphi_b - \frac{3\pi^2 - 14}{36\varphi_b^2} - \frac{3}{2}\varphi^2 - \frac{1}{3}\ln \varphi - \frac{1}{27\varphi^2}, \quad (38)$$

neglecting uncalculated terms going as higher inverse powers of φ_b^2 and both calculated and uncalculated terms going as higher inverse powers of φ^2 . From this expression, one can see that at the end of inflation,

$$\alpha_e \approx \frac{3}{2}\varphi_b^2 - \frac{2}{3}\ln \varphi_b - \frac{3\pi^2 - 14}{36\varphi_b^2} + \text{const.} \quad (39)$$

and the number of e-folds of inflation is

$$N(\varphi_b) = \beta_e = \alpha_e - \alpha_b \approx \frac{3}{2}\varphi_b^2 + \frac{1}{3}\ln \varphi_b - \frac{3\pi^2 - 14}{36\varphi_b^2} + \text{const.}, \quad (40)$$

but, so far as I can see, the numerical constant in this expression cannot be determined by a closed-form expression but requires numerical integration to the end of inflation at $\varphi = -\sqrt{2}\dot{\varphi}$, which is beyond the validity of the slow-roll approximation used above that applies for $\varphi \gg -\sqrt{2}\dot{\varphi}$.

4 Numerical results for the inflationary regime

Since the closed-form approximate expressions derived above do not apply near the end of the inflationary regime, I used Maple to get fairly precise numerical expressions of how many e-folds $N(\varphi_b)$ of inflation occur (the increase in the logarithmic scale factor $\alpha = \ln(ma)$ during the inflationary period that is defined as the initial period during which the second time derivative of the scale factor, \ddot{a} , is positive), and of what the asymptotic value $M_\infty(\varphi_b)$ of the dimensionless ‘mass’ M is, as functions of the initial value φ_b of the dimensionless inflaton scalar field $\varphi \equiv \sqrt{4\pi G/3}\phi$ here written in terms of the physical inflaton scalar field ϕ .

I integrated the equations of evolution from the bounce to the end of inflation for several values of φ_b and found that for $\varphi_b \gtrsim 10$,

$$N(\varphi_b) \approx \frac{3}{2}\varphi_b^2 + \frac{1}{3}\ln \varphi_b - 1.0653 - \frac{3\pi^2 - 14}{36\varphi_b^2} - \frac{0.4}{\varphi_b^4}. \quad (41)$$

I also found that at the end of inflation for $\varphi_b \gtrsim 3$, $\varphi \approx 0.4121$ and $\dot{\varphi} \approx -0.2914$, about one-eighth of the way from its slow-roll value of $-1/3$ to zero. From this one can also deduce that at the end of inflation, $M = M_e \approx 0.2547 \varphi_b^{-3} e^{3N(\varphi_b)}$.

The next question is the φ_b -dependent value of $M_\infty(\varphi_b)$, the asymptotic value of the total rationalized dimensionless ‘mass’ $M = (\varphi^2 + \dot{\varphi}^2)r^3 = (8\pi G/3)m\rho a^3$, where ρ is the scalar field energy density. To make the definition precise, one could take M_∞ to be the value of M at infinite time if the cosmological constant is positive and if the solution expands forever, and to be the value of the dimensionless scale factor $r = ma$ (or, more precisely, of $r(1 - \lambda r^2)$ if λ were not negligible as it is in practice) at the first maximum of r if the universe does not expand forever (which will necessarily be the case if the cosmological constant is not positive). However, in practice, the dimensionless cosmological constant $\lambda \equiv \Lambda/(3m^2) \approx 5.0 \times 10^{-111}$ is so tiny that it is insignificant during the numerical integrations of the inflationary regime, and for large φ_b the maximum $r \sim \exp(4.5\varphi_b^2)$ before the universe would recollapse in the absence of a positive cosmological constant is so huge that one cannot take the numerical integrations that far. Therefore, I shall approximate $M_\infty(\varphi_b)$ by the value M settles down toward in the ‘dust’ regime after the end of inflation but long before one needs to consider the effects of either λ or the spatial curvature $e^{-2\alpha}$.

Numerically, it is still a bit tricky to get precise values for $M_\infty(\varphi_b)$, because $M(t)$ oscillates along with φ (at twice the frequency and at the harmonics of that frequency, since $M(t)$ depends only on $\varphi^2(t)$ and $\dot{\varphi}^2(t)$), with oscillation magnitudes of the basic frequency and its harmonics that decay only as inverse powers of the scale factor. However, one can derive that the following function eliminates the first several harmonics and after the end of inflation rapidly settles down very near its asymptotic value $M_\infty(\varphi_b)$:

$$M_{\text{asym}}(t) = e^{3\alpha} \left\{ (\varphi^2 + \dot{\varphi}^2) + 3\dot{\alpha}\varphi\dot{\varphi} + \frac{9}{32} \left[9(\varphi^2 + \dot{\varphi}^2)^2 - 8\dot{\varphi}^4 + \dot{\alpha}\varphi\dot{\varphi}(3\varphi^2 + \dot{\varphi}^2) \right] \right. \\ \left. + \frac{81}{128} \left[10(\varphi^2 + \dot{\varphi}^2)^3 - 15\varphi^2\dot{\varphi}^4 - 11\dot{\varphi}^6 + 5\dot{\alpha}\varphi\dot{\varphi}(6\varphi^4 + 4\varphi^2\dot{\varphi}^2 - 3\dot{\varphi}^4) \right] \right\}. \quad (42)$$

My numerical results gave

$$M_\infty(\varphi_b) \approx 0.1815\varphi_b^{-3}e^{3N(\varphi_b)} \approx \frac{0.08914e^{4.5\varphi_b^2}}{12\varphi_b^2 + 3\pi^2 - 14 + 24/\varphi_b^2}. \quad (43)$$

One can see that $M_\infty(\varphi_b) \approx 0.7125M_e$, 71% of the value of M at the end of inflation, because of the decaying oscillations of $M(t)$ after the end of inflation.

One can now use this formula along with the criterion of the rightmost inequality of Eq. (19) to deduce that for inflationary solutions starting on Segment 5 with $\lambda = 5 \times 10^{110}$, one needs $\varphi_b \gtrsim 5.4646$ or $\phi_b \gtrsim 2.6700 G^{-1/2}$ or $N(\varphi_b) \gtrsim 44.28$ e-folds of inflation to avoid eventual recollapse and instead have expansion forever in an asymptotic de Sitter regime. This is assuming that the simple inflaton- Λ model applied for all time. In a more realistic model in which the energy of the inflaton field converted to radiation shortly after the end of inflation, one would need a larger initial inflaton field value φ_b and more e-folds of inflation to avoid eventual collapse. For example, if one had all the energy of the inflaton field convert to radiation right at the end of inflation and the universe evolve thereafter as a

radiation- Λ model thereafter, one would need $16\lambda M_e r_e > 1$, which by using the formula above for $M_\infty(\varphi_b)$ and the relation $M_\infty(\varphi_b) \approx 0.7125 M_e$ gives $\varphi_b \gtrsim 6.6069$ or $\phi_b \gtrsim 3.2282 G^{-1/2}$ or $N(\varphi_b) \gtrsim 65.03$ to avoid eventual recollapse. It is rather remarkable that despite the extremely tiny value of λ , the critical initial values of the inflaton field ϕ_b are within one-half an order of magnitude of being unity, essentially because of the very rapid growth of $M_\infty(\varphi_b)$ with φ_b .

Another asymptotic constant late in the ‘dust’ regime (but before either the cosmological constant term λ or the spatial curvature term $e^{-2\alpha}$ becomes important) is the asymptotic value of a certain phase θ . At late times in the ‘dust’ regime, ignoring λ and $e^{-2\alpha}$, one can write $\varphi = \dot{\alpha} \cos \psi$ and $\dot{\varphi} = -\dot{\alpha} \sin \psi$ to define an evolving phase angle ψ , and then the asymptotically constant phase is

$$\theta = \frac{2}{3\dot{\alpha}} - \psi + \sin \psi \cos \psi. \quad (44)$$

(There are more complicated formulas that I have derived for the asymptotically constant phase in the de Sitter phase and/or when the spatial curvature is not negligible, but I shall leave them for a later paper.) Preliminary numerical calculations suggest that the asymptotic value of θ , say $\theta_\infty(\varphi_b)$, is roughly 1.978 for large φ_b , but I have not had time to confirm this and to investigate the dependence on φ_b .

For solutions of our system of a $k = +1$ Friedmann-Robertson-Walker universe with a minimally coupled massive scalar field and a positive cosmological constant that have a bounce at a minimal value of the scale factor and then expand forever in an asymptotically de Sitter phase, there will be an analytic map (not known explicitly, of course) from the initial values at the bounce of φ and $\dot{\varphi}$, say φ_b and $\dot{\varphi}_b$, to the asymptotic values $M_\infty(\varphi_b, \dot{\varphi}_b)$ and $\theta_\infty(\varphi_b, \dot{\varphi}_b)$ (or more precisely, to $M_\infty(\varphi_b, \dot{\varphi}_b)$ and the complex constant $C_\infty(\varphi_b, \dot{\varphi}_b) = e^{i\theta_\infty(\varphi_b, \dot{\varphi}_b)}$, since $\theta_\infty(\varphi_b, \dot{\varphi}_b)$ is actually only defined modulo 2π , but for simplicity I shall continue to refer to $\theta_\infty(\varphi_b, \dot{\varphi}_b)$). For the symmetric-bounce solutions, the solution-space is just one-dimensional (governed by the one parameter φ_b) rather than two-dimensional, with the restriction $\dot{\varphi}_b = 0$, so both M_∞ and θ_∞ are then functions just of φ_b . Hence for these symmetric-bounce solutions, in principle one gets a particular analytic relation $\theta_\infty = \theta_{\infty, \text{sb}}(M_\infty)$.

For the complex solutions of the same minisuperspace system corresponding to the no-boundary proposal [34], one should get a slightly different analytic relation $\theta_\infty = \theta_{\infty, \text{nb}}(M_\infty)$, though one would expect these two functions to approach the same values for very large M_∞ . In the no-boundary case, in which the one free parameter is the complex initial value of φ , say $\varphi(0)$, both $M_\infty(\varphi(0))$ and $\theta_\infty(\varphi(0))$ would be complex for generic complex $\varphi(0)$, but one could choose a one-real-parameter contour in the complex- $\varphi(0)$ plane that would make $M_\infty(\varphi(0))$, say, real. But it would still be the case that even for real $M_\infty(\varphi(0))$, the corresponding $\theta_\infty(\varphi(0))$ would not be quite real, so $\theta_{\infty, \text{nb}}(M_\infty)$ would not be precisely real for real M_∞ as $\theta_{\infty, \text{sb}}(M_\infty)$ is for the symmetric-bounce solutions, as it always is for a real one-parameter set of Lorentzian spacetimes of the FRW form being assumed here. Therefore, it is a bit ambiguous what real Lorentzian solutions correspond to the no-boundary proposal, even asymptotically, since for the complex extrema obeying the no-boundary conditions, one cannot have the two asymptotic constants M_∞ and θ_∞ both real. One

can of course make *ad hoc* choices, such as taking the real Lorentzian solutions that corresponds to real values of M_∞ and then to the real values $\theta_\infty = \text{Re}(\theta_{\infty,\text{sb}}(M_\infty))$ that are the real parts of the complex values $\theta_{\infty,\text{sb}}$ given by the no-boundary proposal for the real values of M_∞ . However, one does need to make some such *ad hoc* choice before getting precisely real Lorentzian solutions from the no-boundary proposal.

5 Inhomogeneous and/or anisotropic perturbations

The symmetric-bounce proposal for the quantum state of the universe is that the universe has inhomogeneous and anisotropic quantum perturbations about the set of classical inflationary solutions described above that are in their ground state at the symmetric bounce hypersurface. In particular, the quantum state of the perturbations on that hypersurface is proposed to be the same as that of the de Sitter-invariant Bunch-Davies vacuum [98] on a de Sitter spacetime with the same radius of the throat as that of the classical background symmetric-bounce inflationary solution at its throat.

Of course, once the massive scalar inflaton field starts to roll down its quadratic potential, the background spacetime will deviate from de Sitter spacetime, so that the quantum perturbations will no longer remain in a de Sitter-invariant state. One would expect the usual inflationary picture of parametric amplification that would result in each inhomogeneous mode leaving its initial vacuum state and becoming excited as the wavelength of that mode is inflated past the Hubble scale given by the expansion rate. In this way one would get the usual inflationary production of density perturbations arising from the initial vacuum fluctuations.

This part of the story is similar to the Hartle-Hawking no-boundary proposal [24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34], which also predicts that the inhomogeneous and anisotropic quantum perturbations start off in the de Sitter-invariant Bunch-Davies vacuum (and admittedly predicts this in a slightly less *ad hoc* way than it is proposed in my symmetric-bounce proposal). However, the main difference is that the symmetric-bounce proposal has the more uniform weighting given by Eqs. (13) for the different values of φ_b and hence of the dimensionless bounce radius $r_b = 1/\varphi_b$, rather than being weighted by the exponential of twice the negative action of the Euclidean hemisphere as in the no-boundary proposal. It is this exponential weighting of the no-boundary proposal that apparently leads to the probabilities being enormously dominated by the largest Euclidean hemispheres, those of empty de Sitter spacetime, and hence for observational probabilities dominated by early-time Boltzmann brains (or Boltzmann solar systems, if one excludes the possibility of observers existing without an entire solar system) [41, 42, 43, 44, 45]. By not having these Euclidean hemispheres and their enormously negative Euclidean actions, the slightly more *ad hoc* symmetric-bounce proposal can avoid the huge domination by empty or nearly-empty de Sitter spacetimes that seems very strongly at odds with our observations of significant structure far beyond ourselves, such as stars.

6 Conclusions

The symmetric-bounce proposal is that the quantum state of the universe is a pure state that consists of a uniform distribution (in a metric induced from the DeWitt metric on the superspace) of components (of different bounce sizes) that each have the quantum fluctuations initially (at the bounce) in their ground state at a moment of time symmetry for a bounce of minimal three-volume. The background spacetimes of this proposal (ignoring the quantum fluctuations) consist of a one-parameter family (at least for one inflaton field; if there are more, there would be as many parameters as bounce values of all the inflaton fields) of time-symmetric inflationary Friedmann-Robertson-Walker universes. For each member of this family, the quantum state of the inhomogeneous and/or anisotropic fluctuations are, at the bounce, the same as the de Sitter-invariant Bunch-Davies vacuum for a de Sitter spacetime with the same curvature as the background FRW universe at its bounce. The entire quantum state is a coherent superposition of all these FRW spacetimes with their quantum fluctuations, with weights given by the DeWitt metric for the bounce configurations.

This symmetric-bounce quantum state reproduces all the usual predictions of inflation but avoids the huge negative Euclidean actions of the Hartle-Hawking no-boundary proposal that seems to make the probabilities dominated by nearly-empty de Sitter spacetime and make our observations of distant structures (e.g., stars) extremely improbable.

It is interesting that since the background inflationary FRW cosmologies for each macroscopic component of the symmetric-bounce quantum state are time symmetric about a bounce, there is actually no big bang or other initial singularity in this model. The classical background universes contract down to the bounce without becoming singular, and then they re-expand in a time-symmetric way. However, because the quantum fluctuations are in their ground state at the bounce, that is the moment of minimal coarse-grained entropy, so entropy grows away from the bounce in both directions of time. Any thermodynamic observer would sense that the arrow of time (given by the observer's memories and observations of the increase of entropy) is increasing away from the bounce, so it would regard the bounce as in its past. Thus one would get the observed time asymmetry of the universe without any of the background classical components having this asymmetry in a global sense. In Wheeleresque terms, the universe would have time-asymmetry without time-asymmetry.

Acknowledgments

I appreciated the hospitality of the Mitchell family and of the George P. and Cynthia W. Mitchell Institute for Fundamental Physics and Astronomy of Texas A&M University at a workshop at Cook's Branch Conservancy, where the basic idea for this paper arose while I was kayaking around Firemeadow Lake, and where I had many useful discussions on related issues with the workshop participants, especially

on this particular issue with Jim Hartle and Thomas Hertog. I am thankful to an anonymous referee for suggesting many improvements and added references, and to Bill Unruh and the University of British Columbia for hospitality while these corrections were made. This research was supported in part by the Natural Sciences and Engineering Research Council of Canada.

References

- [1] D. N. Page, “Do Our Observations Depend upon the Quantum State of the Universe?” arXiv:0907.4751 [hep-th].
- [2] D. N. Page, J. Cosmolog. Astropart. Phys. **0810**, 025 (2008), arXiv:0808.0351 [hep-th].
- [3] D. N. Page, Phys. Lett. B **678**, 41-44 (2009), arXiv:0808.0722 [hep-th].
- [4] D. N. Page, J. Cosmolog. Astropart. Phys. **0708**, 008 (2009), arXiv:0903.4888 [hep-th].
- [5] D. N. Page, “Born Again,” arXiv:0907.4152 [hep-th].
- [6] S. Weinberg, Phys. Rev. Lett. **59**, 2607-2610 (1987).
- [7] A. Linde, “Prospects of Inflationary Cosmology,” astro-ph/9610077.
- [8] A. Vilenkin, Phys. Rev. **D56**, 3238-3241 (1997), astro-ph/9703201.
- [9] A. H. Guth, Phys. Rept. **333**, 555-574 (2000), astro-ph/0002156.
- [10] W. James, *The Will to Believe and Other Essays in Popular Philosophy* (Longmans, Green, and Co., 1897; reprinted by Harvard University Press, Cambridge, MA, 1979), p. 43.
- [11] O. Lodge, *Man and the Universe* (George Doran, New York, 1908), p. 58.
- [12] J. Leslie, *Universes* (Routledge, New York and London, 1989).
- [13] M. Gell-Mann, *The Quark and the Jaguar: Adventures in the Simple and the Complex* (W. H. Freeman, New York, 1994), p. 212.
- [14] D. Deutsch, *The Fabric of Reality* (Allen Lane, The Penguin Press, London and New York, 1997).
- [15] F. Dyson, *Imagined Worlds* (Harvard University Press, Cambridge, Mass., 1997).
- [16] M. J. Rees, *Before the Beginning: Our Universe and Others* (Simon and Schuster, New York, 1997).
- [17] B. J. Carr, ed., *Universe or Multiverse?* (Cambridge University Press, Cambridge, 2007).
- [18] A. Vilenkin, Phys. Rev. Lett. **74**, 846-849 (1995), gr-qc/9406010.
- [19] R. Penrose, *A Complete Guide to the Laws of the Universe* (Vintage, London, 2005), p. 784.
- [20] L. Susskind, *The Cosmic Landscape: String theory and the Illusion of Intelligent Design* (Little, Brown, and Co., New York and Boston, 2006).

- [21] R. C. Tolman, *Relativity, Thermodynamics and Cosmology* (Oxford University Press, Oxford, 1934).
- [22] P. C. W. Davies, *The Physics of Time Asymmetry* (Surrey University Press, Surrey, 1974).
- [23] R. Penrose, in *General Relativity: An Einstein Centenary Survey*, edited by S. W. Hawking and W. Israel (Cambridge University Press, Cambridge, 1979), pp. 581-638.
- [24] S. W. Hawking, in *Astrophysical Cosmology: Proceedings of the Study Week on Cosmology and Fundamental Physics*, edited by H. A. Brück, G. V. Coyne and M. S. Longair (Pontificiae Academiae Scientiarum Scripta Varia, Vatican, 1982), pp. 563-574.
- [25] J. B. Hartle and S. W. Hawking, Phys. Rev. **D28**, 2960-2975 (1983).
- [26] S. W. Hawking, Nucl. Phys. **B239**, 257-276 (1984); in *Relativity, Groups and Topology II*, edited by B. S. DeWitt and R. Stora (North-Holland, 1984), pp. 333-379; Phys. Rev. **D32**, 2489-2495 (1985); in *Field Theory, Quantum Gravity and Strings, Proceedings of the Seminar Series, Meudon and Paris, France, 1984-1985*, edited by H. J. De Vega and N. Sanchez (Lecture Notes in Physics Vol. 246) (Springer, New York, 1986), pp. 1-45; Phys. Scripta **T15**, 151-153 (1987).
- [27] J. J. Halliwell and S. W. Hawking, Phys. Rev. **D31**, 1777-1791 (1985).
- [28] D. N. Page, Phys. Rev. **D32**, 2496-2499 (1985); Phys. Rev. **D34**, 2267-2271 (1986); in *Quantum Concepts in Space and Time*, edited by R. Penrose and C. J. Isham (Clarendon Press, Oxford, 1986), p. 274-285; in *Gravitation: A Banff Summer Institute*, edited by R. B. Mann and P. Wesson (World Scientific, Singapore, 1991), p. 135-170.
- [29] J. J. Halliwell, in *Quantum Cosmology and Baby Universes*, edited by S. Coleman, J. Hartle, T. Piran, and S. Weinberg (World Scientific, Singapore, 1991), p. 159-243; Sci. Am. **265**, No. 6, 28-35 (1991).
- [30] S. W. Hawking, R. Laflamme, and G. W. Lyons, Phys. Rev. **D47**, 5342-5356 (1993), gr-qc/9301017.
- [31] S. W. Hawking and T. Hertog, Phys. Rev. **D66**, 123509 (2002), hep-th/0204212.
- [32] D. N. Page, in *The Future of Theoretical Physics and Cosmology: Celebrating Stephen Hawking's 60th Birthday*, edited by G. W. Gibbons, E. P. S. Shellard, and S. J. Rankin (Cambridge University Press, Cambridge, 2003), pp. 621-648, hep-th/0610121.
- [33] S. W. Hawking and T. Hertog, Phys. Rev. **D73**, 123527 (2006), hep-th/0602091.
- [34] J. B. Hartle, S. W. Hawking, and T. Hertog, Phys. Rev. Lett. **100**, 201301 (2008), arXiv:0711.4630 [hep-th]; Phys. Rev. D **77**, 123537 (2008), arXiv:0803.1663 [hep-th].

- [35] A. Vilenkin, Phys. Lett. **117B**, 25-28 (1982); Phys. Rev. **D27**, 2848-2855 (1983); Phys. Rev. **D30**, 509-511 (1984); Nucl. Phys. **B252**, 141-151 (1985); Phys. Rev. **D33**, 3560-3569 (1986); Phys. Rev. **D37**, 888-897 (1988); Phys. Rev. **D39**, 1116-1122 (1989); Phys. Rev. **D50**, 2581-2594 (1994), gr-qc/9403010; Phys. Rev. **D58**, 067301 (1998), gr-qc/9804051; in *The Future of Theoretical Physics and Cosmology: Celebrating Stephen Hawking's 60th Birthday*, edited by G. W. Gibbons, E. P. S. Shellard, and S. J. Rankin (Cambridge University Press, Cambridge, 2003), pp. 649-666; *Many Worlds in One: The Search for Other Universes* (Hill and Wang, New York, 2006).
- [36] A. D. Linde, Zh. Eksp. Teor. Fiz. **87**, 369-374 (1984) [Sov. Phys. JETP **60**, 211-213 (1984)]; Lett. Nuovo Cimento **39**, 401-405 (1984); Phys. Scripta **T36**, 30-54 (1991); AIP Conf. Proc. **478**, 30-37 (1999).
- [37] Ya. B. Zel'dovich and A. A. Starobinsky, Pis'ma Astron. Zh. **10**, 323-328 (1984) [Sov. Astron. Lett. **10**, 135-137 (1984)].
- [38] V. A. Rubakov, Phys. Lett. **148B**, 280-286 (1984).
- [39] T. Vachaspati and A. Vilenkin, Phys. Rev. **D37**, 898-903 (1988).
- [40] J. Garriga and A. Vilenkin, Phys. Rev. **D56**, 2464-2468 (1997), gr-qc/9609067.
- [41] L. Susskind, private communication (2002).
- [42] L. Dyson, M. Kleban, and L. Susskind, J. High Energy Phys. **0210**, 011 (2002), hep-th/0208013.
- [43] N. Goheer, M. Kleban, and L. Susskind, J. High Energy Phys. **0307**, 056 (2003), hep-th/0212209.
- [44] L. Susskind, in *Universe or Multiverse?*, edited by B. J. Carr (Cambridge University Press, Cambridge, 2007), pp. 247-266, hep-th/0302219.
- [45] D. N. Page, J. Cosmolog. Astropart. Phys. **0701**, 004 (2007), hep-th/0610199.
- [46] S. Perlmutter *et al.* Astrophys. J. **483**, 565-581 (1997), astro-ph/9608192; Nature **391**, 51-54 (1998), astro-ph/9712212; Astrophys. J. **517**, 565-586 (1999), astro-ph/9812133.
- [47] A. G. Riess *et al.* Astronom. J. **116**, 1009-1038 (1998), astro-ph/9805201; Astrophys. J. **560**, 49-71 (2001), astro-ph/0104455; Astrophys. J. **607**, 665-687 (2004), astro-ph/0402512.
- [48] S. Perlmutter, M. S. Turner, and M. J. White, Phys. Rev. Lett. **83**, 670-673 (1999), astro-ph/9901052.
- [49] J. L. Tonry *et al.* Astrophys. J. **594**, 1-24 (2003), astro-ph/0305008.
- [50] D. N. Spergel *et al.* Astrophys. J. Suppl. **148**, 175-194 (2003), astro-ph/0302209; astro-ph/0603449.

- [51] M. Tegmark *et al.* Phys. Rev. **D69**, 103501 (2004), astro-ph/0310723.
- [52] P. Astier *et al.* Astron. Astrophys. **447**, 31-48 (2006), astro-ph/0510447.
- [53] R. Bousso and S. W. Hawking, Phys. Rev. **D54**, 6312-6322 (1996), gr-qc/9606052.
- [54] A. D. Linde, Phys. Rev. **D58**, 083514 (1998), gr-qc/9802038.
- [55] S. W. Hawking and N. G. Turok, "Comment on 'Quantum Creation of an Open Universe,' by Andrei Linde," gr-qc/9802062.
- [56] N. G. Turok and S. W. Hawking, Phys. Lett. **B432**, 271-278 (1998), hep-th/9803156.
- [57] A. Vilenkin, Phys. Rev. **D58**, 067301 (1998), gr-qc/9804051; in *Conference on Particle Physics and the Early Universe (COSMO 98), Monterey, CA, 15-20 Nov 1998*, edited by D. O. Caldwell (American Institute of Physics, New York, 1999), pp. 23-29, gr-qc/9812027.
- [58] D. N. Page, Class. Quant. Grav. **25**, 154011 (2008), arXiv:0707.2081 [hep-th].
- [59] S. B. Giddings and D. Marolf, Phys. Rev. D **76**, 064023 (2007), arXiv:0705.1178 [hep-th].
- [60] A. Albrecht, in *Science and Ultimate Reality: Quantum Theory, Cosmology, and Complexity*, edited by J. D. Barrow, P. C. W. Davies, and C. L. Harper, Jr. (Cambridge University Press, Cambridge, 2004), pp. 363-401, arXiv:astro-ph/0210527.
- [61] A. Albrecht and L. Sorbo, Phys. Rev. D **70**, 063528 (2004), arXiv:hep-th/0405270.
- [62] D. N. Page, J. Korean Phys. Soc. **49**, 711-714 (2006), arXiv:hep-th/0510003.
- [63] A. V. Yurov and V. A. Yurov, "One More Observational Consequence of Many-Worlds Quantum Theory," arXiv:hep-th/0511238.
- [64] D. N. Page, Phys. Rev. D **78**, 063535 (2008), arXiv:hep-th/0610079.
- [65] R. Bousso and B. Freivogel, J. High Energy Phys. **0706**, 018 (2007), arXiv:hep-th/0610132.
- [66] D. N. Page, J. Cosmolog. Astropart. Phys. **0701**, 004 (2007), arXiv:hep-th/0610199.
- [67] A. Linde, J. Cosmolog. Astropart. Phys. **0701**, 022 (2007), arXiv:hep-th/0611043.
- [68] D. N. Page, Phys. Rev. D **78**, 063536 (2008), arXiv:hep-th/0611158.
- [69] A. Vilenkin, J. High Energy Phys. **0701**, 092 (2007), arXiv:hep-th/0611271.

- [70] D. N. Page, Phys. Lett. B **669**, 197-200 (2008), arXiv:hep-th/0612137.
- [71] V. Vanchurin, Phys. Rev. D **75**, 023524 (2007) [arXiv:hep-th/0612215].
- [72] T. Banks, “Entropy and Initial Conditions in Cosmology,” arXiv:hep-th/0701146.
- [73] S. Carlip, J. Cosmolog. Astropart. Phys. **0706**, 001 (2007), arXiv:hep-th/0703115.
- [74] J. B. Hartle and M. Srednicki, Phys. Rev. D **75**, 123523 (2007), arXiv:0704.2630.
- [75] S. B. Giddings, Mod. Phys. Lett. A **22**, 2949-2954 (2007), arXiv:0705.2197 [hep-th].
- [76] D. N. Page, “Typicality Defended,” arXiv:0707.4169 [hep-th].
- [77] M. Li and Y. Wang, “Typicality, Freak Observers and the Anthropic Principle of Existence,” arXiv:0708.4077 [hep-th].
- [78] D. N. Page, in *Proceedings of 13th International Congress of Logic, Methodology and Philosophy of Science*, edited by C. Glymour, W. Wang, and D. Westerståhl (Kings College Publications, 2009), arXiv:0712.2240 [hep-th].
- [79] R. Bousso, Gen. Rel. Grav. **40**, 607-637 (2008), arXiv:0712.3324 [hep-th].
- [80] R. Bousso, B. Freivogel, and I-S. Yang, Phys. Rev. D **77**, 103514 (2008), arXiv:0712.3324 [hep-th].
- [81] N. Arkani-Hamed, S. Dubovsky, L. Senatore, and G. Villadoro, J. High Energy Phys. **0803**, 075 (2008), arXiv:0801.2399 [hep-ph].
- [82] J. R. Gott, III, “Boltzmann Brains: I’d Rather See than Be One,” arXiv:0802.0233 [gr-qc].
- [83] D. N. Page, Phys. Rev. D **78**, 023514 (2008), arXiv:0804.3592 [hep-th].
- [84] B. Freivogel and M. Lippert, J. High Energy Phys. **0812**, 096 (2008), arXiv:0807.1104 [hep-th].
- [85] D. N. Page, “Possible Anthropic Support for a Decaying Universe: A Cosmic Doomsday Argument,” arXiv:0907.4153 [hep-th].
- [86] D. N. Page, Class. Quant. Grav. **1**, 417-427 (1984).
- [87] B. S. DeWitt, Phys. Rev. **160**, 11131148 (1967).
- [88] W. L. Freedman *et al.*, Astrophys. J. **553**, 47-72 (2001), astro-ph/0012376.
- [89] A. Linde, *Particle Physics and Inflationary Cosmology* (Harwood Academic Publishers, Chur, Switzerland, 1990).

- [90] A. R. Liddle and D. H. Lyth, *Cosmological Inflation and Large-Scale Structure* (Cambridge University Press, Cambridge, 2000).
- [91] G. W. Gibbons, S. W. Hawking, and J. M. Stewart, Nucl. Phys. B **281**, 736-751 (1987).
- [92] H. D. Conradi and H. D. Zeh, Phys. Lett. A **154**, 321-326 (1991).
- [93] A. O. Barvinsky and A. Y. Kamenshchik, Class. Quant. Grav. **7**, L181-L186 (1990).
- [94] A. O. Barvinsky, A. Y. Kamenshchik and I. P. Karmazin, Annals Phys. **219**, 201-242 (1992).
- [95] A. O. Barvinsky and A. Y. Kamenshchik, Phys. Lett. B **332**, 270-276 (1994), arXiv:gr-qc/9404062.
- [96] A. A. Starobinsky, Sov. Astron. Lett. **4**, 82 (1978).
- [97] A. A. Starobinsky, "Spectrum of Initial Perturbations in Open and Closed Inflationary Models," astro-ph/9603075.
- [98] T. S. Bunch and P. C. W. Davies, Proc. Roy. Soc. Lond. A **360**, 117-134 (1978).